

**An  $n = (1, 1)$  super-Toda Model Based on  $OSp(1|4)$** **Dmitrij Sorokin\* and Francesco Toppan,***Dipartimento di Fisica “Galileo Galilei”,  
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toppan@pd.infn.it***Abstract**

We show that a Hamiltonian reduction of affine Lie superalgebras having bosonic simple roots (such as  $OSp(1|4)$ ) *does* produce supersymmetric Toda models, with superconformal symmetry being nonlinearly realised for those fields of the Toda system which are related to the bosonic simple roots of the superalgebra. A fermionic  $b - c$  system of conformal spin  $(\frac{3}{2}, -\frac{1}{2})$  is a natural ingredient of such models.

PACS: 11-17.+y; 11.30.Pb,Qc; 11.40.-q

Keywords: Affine superalgebras, Hamiltonian reduction, Toda models, supersymmetry, superstrings.

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# 1 Introduction

A systematic way of getting exactly-solvable Toda models [1] is to carry out a Hamiltonian reduction of affine Lie algebras or Wess–Zumino–Novikov–Witten models associated with them [2, 3]. The Hamiltonian reduction consists in imposing (first-class) constraints on components of algebra-valued currents. This procedure is of particular importance for understanding the underlying algebraic structure of the Toda models (such as  $W$ -algebra extension of  $d = 2$  conformal symmetry), for simplifying the construction of the general solution of the Toda equations and solving the quantization problem (see [3] for a review).

The Hamiltonian reduction procedure is also applicable to the case of affine superalgebras and corresponding super-WZNW models [4]. On one hand it serves, for instance, as a powerful method for studying superconformal and  $W$ -structure of superstring theory [5, 6], and on the other hand, as a result of the Hamiltonian reduction one gets super-Toda models and super- $W$ -algebras [4, 7]. However, in contrast to the bosonic case not all superalgebras have been involved into the production of *supersymmetric* Toda models yet. This is connected with a well-known fact that the standard Hamiltonian reduction (which implies imposing (first-class) constraints directly on basic (super)currents of the affine (super)algebras) leads to Toda models with explicitly broken two-dimensional supersymmetry if the affine superalgebras subject to the Hamiltonian reduction contain not only odd (or fermionic) simple roots but also even (bosonic) simple roots in any simple root system of their generators [7]. This also takes place in the case of the bosonic super-affine Lie algebras, i.e. algebras whose currents are superfields in a  $d = 2$  superspace and whose root systems are always bosonic.

In [8] it was demonstrated how one can generalize the Hamiltonian reduction procedure to get super-Toda-like equations from super-WZNW models based on bosonic Lie groups. The simplest example of an  $n = (1, 1)$  supersymmetric  $Sl(2, \mathbf{R})$  WZNW model was considered in detail, and supersymmetric version of the Liouville equation alternative to the standard one [9] was obtained by imposing nonlinear constraints on  $Sl(2, \mathbf{R})$  supercurrents. Such a Liouville system emerged before as a system of equations for a single bosonic (Liouville) and two fermionic physical degrees of freedom of a classical Green–Schwarz superstring propagating in  $N = 2$ ,  $D = 3$  flat target superspace [10]<sup>1</sup>. Worldsheet  $n = (1, 1)$  superconformal symmetry of the system is nonlinearly realized on the bosonic Liouville field and the two fermionic fields, the latter transform as Goldstone fermions and have negative conformal spin  $-\frac{1}{2}$  unusual for matter fields. This means that supersymmetry is broken spontaneously. The corresponding stress-tensor of conformal spin 2 and the supersymmetry current of spin  $3/2$  are constructed out of the spin 0 Liouville field, the spin  $-\frac{1}{2}$  field and its conjugate momentum of spin  $\frac{3}{2}$  (see [8] for the details). Such a so called fermionic  $b$ – $c$  system plays a particular role in superconformal theory and,

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<sup>1</sup>One should not confuse this Liouville system describing classical physical modes of the string with (super)Liouville modes arising as anomalies of quantized noncritical strings [11].

for instance, in constructing a theory of embedding fermionic strings [12, 5, 13]. Thus, the generalization of the Hamiltonian reduction in application to (super)affine (super)algebras with simple bosonic roots allows one to naturally obtain (already at the classical level and without untwisting ghosts [13]) superconformal  $b - c$  systems which might correspond to unexplored vacua of string theory. In [5] an alternative reduction procedure was used to get  $n = 2$  superconformal  $b - c$  structures and realizations of super- $W_n$  algebras from the  $Sl(n|n - 1)$  superalgebras containing simple bosonic roots in a Gauss decomposition.

In the present article we consider the Hamiltonian reduction of a current superalgebra in  $n = (1, 1)$ ,  $d = 2$  superspace based on the rank 2 superalgebra  $OSp(1|4)$ , which is the simplest example of the superalgebras with simple bosonic roots (its simple root system consists of one bosonic and one fermionic simple root [14, 15]). Note that Hamiltonian reduction of affine  $OSp(1|4)$  currents in  $d = 2$  bosonic space, which results in a nonsuper-symmetric  $W$ -algebra structure, has been carried out in [16]. Below we shall show how, by imposing appropriate nonlinear constraints on  $n = (1, 1)$  supercurrents corresponding to the bosonic simple roots, one can circumvent the obstacle caused by the presence of the bosonic simple roots [7] and get an  $n = (1, 1)$  supersymmetric Toda model with superconformal symmetry being nonlinearly realized for those components of the Toda system which are related to the bosonic simple root of  $OSp(1|4)$ .

The set of fields of the super-Toda model consists of two bosonic fields of conformal spin 0, which are coupled to two fermionic fields of spin  $\frac{1}{2}$  and two (left-right) chiral Goldstone fermions of spin  $-\frac{1}{2}$ . These fields are accompanied by a decoupled pair of free left-right chiral fermions of spin  $\frac{3}{2}$ . The spin  $\frac{3}{2}$  and  $-\frac{1}{2}$  fields form the fermionic  $b - c$  system. In the absence of the fermions the bosonic fields obey the equations of motion of a Toda model based on  $Sp(4)$ .

All the constraints imposed on  $OSp(1|4)$  supercurrent components are of the first class except one fermionic second-class constraint. A complete gauge fixing of local symmetries corresponding to the first-class constraints results in a set of five two-dimensional holomorphic and antiholomorphic current fields which survive the Hamiltonian reduction. These are (in the holomorphic sector) a conformal spin  $-\frac{1}{2}$  field  $c$ , spin  $\frac{3}{2}$  field  $G$ , spin 2 field  $T$ , spin  $\frac{5}{2}$  field  $W_{\frac{5}{2}}$  and spin 4 field  $W_4$ . The currents  $G$  and  $T$  generate  $n = 1$  superconformal symmetry of the Toda model, while entire set of the currents might generate a super- $W$ -algebra. We shall discuss this point in Conclusion upon establishing links (via truncation) of our  $OSp(1|4)$  model with the Hamiltonian reduction of affine  $OSp(1|4)$  currents in  $d = 2$  bosonic space [16], with the supersymmetric versions of the Liouville model based on the  $OSp(1|2)$  [9] and  $Sl(2, \mathbf{R})$  [10], and with a bosonic Toda model based on  $Sp(4)$  [2].

## 2 Affine $OSp(1|4)$ superalgebra

$OSp(1|4)$  is a 14-dimensional superalgebra of rank 2. It has 10 bosonic and 4 fermionic generators. Two bosonic generators  $H_\alpha$  form a Cartan subalgebra (index  $\alpha = 1, 2$  or denote  $b$  and  $f$  which stand, respectively, for “bosonic” and “fermionic”). The  $OSp(1|4)$  root system consists of one positive (negative) fermionic simple root and one positive (negative) bosonic simple root. We shall denote fermionic and bosonic generators associated with the simple roots, respectively,  $E_{\pm f}$  and  $E_{\pm b}$ . The remaining generators (collectively denoted as  $E_{+i}$ ,  $E_{-i}$  ( $i = 1, 2, 3, 4$ )) correspond to higher positive and negative roots.

The (anti)commutation properties of the  $OSp(1|4)$  generators can be derived by use of the fundamental representation of  $OSp(1|4)$  realized on  $(4+1) \times (4+1)$  supermatrices (with 4 bosonic and 1 fermionic indices)

$$\begin{aligned} H_1 &= e_{11} - e_{22} - e_{33} + e_{44}; & H_2 &= e_{33} - e_{44} \\ E_{+b} &= e_{13} + e_{42}; & E_{-b} &= e_{24} + e_{31} \\ E_{+f} &= e_{35} + e_{54}; & E_{-f} &= e_{53} - e_{45} \\ \\ E_{+1} &= 2e_{34}; & E_{-1} &= -2e_{43} \\ E_{+2} &= e_{15} - e_{52}; & E_{-2} &= -(e_{51} + e_{21}) \\ E_{+3} &= 2(e_{14} - e_{32}); & E_{-3} &= 2(e_{41} - e_{23}) \\ E_{+4} &= -4e_{12}; & E_{-4} &= 4e_{21} \end{aligned}$$

In the above formulas we denote with  $e_{ij}$  the supermatrices having as entries  $c_{kl} = \delta_{ik}\delta_{jl}$ . Notice that discarding the fermionic generators  $E_{\pm f}$ ,  $E_{\pm 2}$  we get the bosonic  $sp(4)$  subalgebra of  $OSp(1|4)$ , which is then realised on bosonic  $(4 \times 4)$  matrices. For our purposes we shall mainly need the form of the  $OSp(1|4)$  Cartan matrix which is given by [15]:

$$K_{\alpha\beta} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}. \quad (1)$$

The holomorphic and antiholomorphic copies of the classical  $n = (1, 1)$ ,  $d = 2$  affine  $OSp(1|4)$  superalgebra are generated by fermionic supercurrents

$$\Psi(Z) = \Psi^{0\alpha} H_\alpha + \Psi^{-f} E_{-f} + \Psi^{-b} E_{-b} + \Psi^{+f} E_{+f} + \Psi^{+b} E_{+b} + \sum_i \Psi^{-i} E_{-i} + \sum_i \Psi^{+i} E_{+i}, \quad (2)$$

$$\bar{\Psi}(\bar{Z}) = \bar{\Psi}^{0\alpha} H_\alpha + \bar{\Psi}^{-f} E_{-f} + \bar{\Psi}^{-b} E_{-b} + \bar{\Psi}^{+f} E_{+f} + \bar{\Psi}^{+b} E_{+b} + \sum_i \bar{\Psi}^{-i} E_{-i} + \sum_i \bar{\Psi}^{+i} E_{+i}.$$

$n = (1, 1)$ ,  $d = 2$  superspace is parametrized by supercoordinates  $Z = (z, \theta)$ ,  $\bar{Z} = (\bar{z}, \bar{\theta})$  (where  $z, \bar{z}$  are bosonic and  $\theta, \bar{\theta}$  are fermionic light-cone coordinates). The supercovariant derivatives  $D = \frac{\partial}{\partial \theta} + i\theta \frac{\partial}{\partial z}$ ,  $\partial = \frac{\partial}{\partial z}$  and  $\bar{D} = \frac{\partial}{\partial \bar{\theta}} + i\bar{\theta} \frac{\partial}{\partial \bar{z}}$ ,  $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$  form the following algebra

$$\{D, D\} = 2i \frac{\partial}{\partial z}, \quad \{\bar{D}, \bar{D}\} = 2i \frac{\partial}{\partial \bar{z}}, \quad \{D, \bar{D}\} = 0. \quad (3)$$

The  $n = (1, 1)$  superconformal transformations of  $Z, \bar{Z}$  and their derivatives are:

$$\begin{aligned} Z' &= Z'(Z), & \bar{Z}' &= \bar{Z}'(\bar{Z}); \\ D' &= e^{-\Lambda} D, & \bar{D}' &= e^{-\bar{\Lambda}} \bar{D}, & \text{where } \Lambda &= \log D\theta'(Z), & \bar{\Lambda} &= \log \bar{D}\bar{\theta}'(\bar{Z}); \end{aligned} \quad (4)$$

and

$$\Psi'(Z') = e^{-\Lambda} \Psi(Z), \quad \bar{\Psi}'(\bar{Z}') = e^{-\bar{\Lambda}} \bar{\Psi}(\bar{Z}).$$

Under (left–right) affine  $OSp(1|4)$  transformations the supercurrents transform as follows:

$$\Psi'(Z) = g_L^{-1}(Z) \Psi g_L(Z) + i g_L^{-1} D g_L; \quad \bar{\Psi}'(\bar{Z}) = g_R(\bar{Z}) \bar{\Psi} g_R^{-1}(\bar{Z}) + i g_R^{-1} \bar{D} g_R. \quad (5)$$

Poisson brackets of the supercurrents are [4]

$$\{str(A\Psi(X)), str(B\Psi(Y))\}_{PB} = -\delta(X-Y) str[A, B] \Psi(Y) - i D_X \delta(X-Y) str(AB), \quad (6)$$

where  $A$  and  $B$  stand for the  $OSp(1|4)$  generators;  $X = (z_1, \theta_1)$ ,  $Y = (z_2, \theta_2)$ ,  $\delta(X-Y) = \delta(z_1 - z_2)(\theta_1 - \theta_2)$ , and  $D_X = \frac{\partial}{\partial \theta_1} + i\theta_1 \frac{\partial}{\partial z_1}$ . It is implied that the Poisson brackets are equal–time, i.e.  $z_1 + \bar{z}_1 = z_2 + \bar{z}_2$ . (The Poisson brackets of  $\bar{\Psi}(\bar{Z})$  have the same form).

By definition the affine  $OSp(1|4)$  supercurrents satisfy the (anti)chirality conditions, which are preserved under the superconformal (4) and superaffine (5) transformations:

$$\bar{D}\Psi(Z) = 0, \quad D\bar{\Psi}(\bar{Z}) = 0. \quad (7)$$

We assume these conditions to arise as the equations of motion of an  $OSp(1|4)$  WZNW model so that  $\Psi(Z)$  and  $\bar{\Psi}(\bar{Z})$  are expressed in terms of an  $OSp(1|4)$  supergroup element  $G(Z, \bar{Z})$ :

$$\Psi = -i D G G^{-1}, \quad \bar{\Psi} = i G^{-1} \bar{D} G. \quad (8)$$

To get super–Toda equations from (7) one should express the supercurrents in terms of superfields of the Gauss decomposition of  $G(Z, \bar{Z})$ :

$$G = e^{\beta^\alpha E_{+\alpha} + \beta^i E_{+i}} e^{\Phi_\alpha H_\alpha} e^{\gamma^\alpha E_{-\alpha} + \gamma^i E_{-i}} \quad (9)$$

and perform a Hamiltonian reduction, i.e. to impose constraints on  $OSp(1|4)$ –valued components of the supercurrents. Note that they can be imposed at a purely algebraic level, i.e. without referring to the equation (8).

Since the form of  $\Psi(Z)$  and  $\bar{\Psi}(\bar{Z})$  components in terms of superfields  $\beta(Z, \bar{Z})$ ,  $\gamma(Z, \bar{Z})$  and  $\Phi(Z, \bar{Z})$  is essentially simplified upon imposing the constraints, we shall first discuss these constraints and then write down expressions for  $\Psi(Z)$  and  $\bar{\Psi}(\bar{Z})$  in terms of the Gauss decomposition superfields which, being substituted into (7), give rise to the super–Toda equations.

The Hamiltonian reduction procedure [2, 3, 4] prescribes constraining holomorphic supercurrents associated with simple negative roots to be nonzero constants, and the ones

associated with the negative nonsimple roots are put equal to zero. (In the antiholomorphic sector the positive-root supercurrent components are constrained).

In the case of superalgebras [7, 4] the Hamiltonian reduction causes no problems with preserving supersymmetry if a root system of the superalgebra contains only simple fermionic roots. Then the corresponding supercurrents (such as  $\Psi^{-f}$  in (2)) are bosonic and one can put them equal to constants without violating supersymmetry. But if, as in the case of  $OSp(1|4)$ , a root system of a superalgebra contains bosonic simple roots, the corresponding supercurrents (such as  $\Psi^{-b}$ ) are fermionic and constraining them to be Grassmann constants or equal to the Grassmann coordinate  $\theta$  [7] one explicitly breaks supersymmetry of the model.

In [8] it was proposed to overcome this problem by constructing  $OSp(1|4)$  bosonic supercurrents out of the fermionic supercurrents as prescribed by the Maurer–Cartan equations (see [8] for details):

$$J(Z) = D\Psi - i\Psi\Psi, \quad \bar{J}(\bar{Z}) = \bar{D}\bar{\Psi} - i\bar{\Psi}\bar{\Psi} \quad (10)$$

(where matrix multiplication is implied) and then put equal to constants the superalgebra-valued components of  $J$  and  $\bar{J}$  corresponding, respectively, to the negative and positive bosonic simple-root generators  $E_{-b}$  and  $E_{+b}$ .

Since the constraints imposed this way have a superfield form, supersymmetry is not explicitly broken but, as we shall see, it is nonlinearly realized on part of the fields which survive the Hamiltonian reduction.

Thus we impose on fermionic supercurrents (7) the following set of constraints:

$$\begin{aligned} \Psi^{-f} &= \mu_f, & \Psi^{-i} &= 0, & J^{-b} &\equiv D\Psi^{-b} + 2i\Psi^{0b}\Psi^{-b} = \mu_b; \\ \bar{\Psi}^{+f} &= \nu_f, & \bar{\Psi}^{+i} &= 0, & \bar{J}^{+b} &\equiv \bar{D}\bar{\Psi}^{+b} - 2i\bar{\Psi}^{0b}\bar{\Psi}^{+b} = \nu_b, \end{aligned} \quad (11)$$

where the factor 2 comes from the Cartan matrix (1), and  $\mu_\alpha$  and  $\nu_\alpha$  ( $\alpha = b, f$  or  $(1,2)$ ) are arbitrary constants.

We postpone the Hamiltonian analysis of these constraints to Section 4 and turn to getting the supersymmetric Toda system associated with  $OSp(1|4)$ .

### 3 $OSp(1|4)$ super–Toda equations.

To derive the  $n = (1,1)$  supersymmetric Toda equations from (7) and (11) one should write down supercurrents  $\Psi^{0\alpha} = (\Psi^{0f}, \Psi^{0b})$ ,  $\Psi^{-\alpha} = (\Psi^{-f}, \Psi^{-b})$  and their antiholomorphic counterparts in terms of fields of the Gauss decomposition (9). With taking into account the constraints we get

$$\begin{aligned} \Psi^{0f} &= -iD\Phi_2 + \mu_f\beta^f, & \Psi^{0b} &= -iD\Phi_1 + \beta^b\Psi^{-b}, \\ \Psi^{-b} &= -iD\gamma^b e^{-K_{b\alpha}\Phi^\alpha}, & \Psi^{-f} &= -iD\gamma^f e^{-K_{f\alpha}\Phi^\alpha} = \mu_f; \end{aligned}$$

$$\begin{aligned}\bar{\Psi}^{0f} &= i\bar{D}\Phi_2 + \nu_f\gamma^f, & \bar{\Psi}^{0b} &= iD\Phi_1 + \gamma^b\bar{\Psi}^{+b}, \\ \bar{\Psi}^{+b} &= i\bar{D}\beta^b e^{-K_{b\alpha}\Phi^\alpha}, & \bar{\Psi}^{+f} &= i\bar{D}\beta^f e^{-K_{f\alpha}\Phi^\alpha} = \nu_f.\end{aligned}\tag{12}$$

In (12) we assume summation over  $\alpha = 1, 2$ , and remember that  $b$  and  $f$  correspond, respectively, to the single bosonic and fermionic simple root and stand for the index 1 and 2 of the Cartan matrix components (1).

Hitting the r.h.s of  $\Psi^{0f}$  and  $\Psi^{0b}$  in (12) with the supercovariant derivative  $\bar{D}$  and taking into account the chirality conditions (7), the form of the Cartan matrix (1) and other expressions in (12) we get the following system of superfield equations:<sup>2</sup>

$$\bar{D}D\Phi_2 = i\mu_f\nu_f e^{\Phi_2-\Phi_1}, \quad \bar{D}D\Phi_1 = e^{2\Phi_1-\Phi_2}\bar{\Psi}^{+b}\Psi^{-b}, \quad \bar{D}\Psi^{-b} = 0 = D\bar{\Psi}^{+b}.\tag{13}$$

To Eqs. (13) one must add the constraints on  $J^{-b}$  and  $\bar{J}^{+b}$  in (11), which now take the form:

$$D\Psi^{-b} + 2D\Phi_1\Psi^{-b} = \mu_b, \quad \bar{D}\bar{\Psi}^{+b} + 2\bar{D}\Phi_1\bar{\Psi}^{+b} = \nu_b.\tag{14}$$

The system of equations (13) and (14) is the  $OSp(1|4)$  super-Toda system we have been looking for.

Let us relate the super-Toda system (13), (14) to some known (super)-Toda equations.

The  $OSp(1|4)$ -based Toda system with explicitly broken  $n = (1, 1)$  supersymmetry [7] is recovered by putting  $\Psi_{-b} = \theta$  and  $\bar{\Psi}_{+b} = \bar{\theta}$ . Notice that it contains less fermionic degrees of freedom than in (13) and (14). If we in addition put equal to zero all higher superfield components of  $\Phi_1$  and  $\Phi_2$  (except the leading ones), we get a bosonic  $Sp(4)$  Toda model [2].

The  $n = (1, 1)$  super-Liouville equation [9] based on an  $OSp(1|2)$  subgroup of  $OSp(1|4)$  is obtained as a truncation of the first equation of (13) by putting in (13) and (14)  $\Psi^{-b} = \bar{\Psi}^{+b} = \mu_b = \nu_b = 0$  and  $\Phi_1 = \text{constant}$ .

The alternative version [10, 8] of the  $n = (1, 1)$  super-Liouville system based on an  $Sl(2, R)$  subgroup of  $OSp(1|4)$  arises as a truncated form of Eqs. (13), (14) where  $\Phi_2 = \text{constant}$  and  $\mu_f = \nu_f = 0$ . (Note that putting some of  $\Phi_\alpha$  equal to constants and  $\mu_\alpha = \nu_\alpha = 0$  is equivalent to putting to zero corresponding supercurrents of  $OSp(1|4)$ , which thus truncates the affine  $OSp(1|4)$  down to its subalgebras).

Now let us discuss in more detail properties of Eqs. (13), (14). The system is invariant under the following  $n = (1, 1)$  superconformal transformations (4) of the superfields:

$$\begin{aligned}\Psi'^{-b}(Z') &= e^\Lambda\Psi^{-b}, & \Psi'^{+b}(Z') &= e^{\bar{\Lambda}}\bar{\Psi}^{+b}, \\ \Phi'_1(Z') &= \Phi_1 - 3(\Lambda + \bar{\Lambda}) - \frac{5}{2}(D\Lambda\Psi^{-b} + \bar{D}\bar{\Lambda}\bar{\Psi}^{+b}), \\ \Phi'_2(Z') &= \Phi_2 - 4(\Lambda + \bar{\Lambda}) - \frac{5}{2}(D\Lambda\Psi^{-b} + \bar{D}\bar{\Lambda}\bar{\Psi}^{+b}).\end{aligned}\tag{15}$$

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<sup>2</sup>It is possible, and indeed preferable, to treat  $\Psi^{-b}$ ,  $\bar{\Psi}^{+b}$  as independent superfields; in this case  $\gamma^b$ ,  $\beta^b$  do not enter the equations (13,14).

One can see that the superconformal properties of the supercurrents  $\Psi$ ,  $\bar{\Psi}$  have changed. This is a consequence of imposing the constraints (11) which break part of affine  $OSp(1|4)$  symmetries [2, 3, 4]. For instance the superaffine transformations corresponding to the  $OSp(1|4)$  Cartan subalgebra are combined with initial superconformal transformations of the supercurrents to produce the modified superconformal transformations (15) which preserve the form of the constraints (11). The remaining local gauge symmetries of the model are generated by those constraints in (11) which are of the first class. We discuss this point in the next Section.

Eqs. (15) tell us that superconformal symmetry is nonlinearly realized on superfields of the super-Toda model. In order that one of the Cartan superfields transforms independently of  $\Psi^{-b}$ ,  $\bar{\Psi}^{+b}$ , it is convenient to redefine the fields as follows:

$$\begin{aligned}\hat{\Phi}_1 &= \frac{\Phi_1}{3} - \frac{2}{3}(D\Phi_1\Psi^{-b} + \bar{D}\Phi_1\bar{\Psi}^{+b}), & \hat{\Phi}'_1(Z') &= \hat{\Phi}_1 - \Lambda - \bar{\Lambda} - \frac{1}{2}(D\Lambda\Psi^{-b} + \bar{D}\bar{\Lambda}\bar{\Psi}^{+b}); \\ \hat{\Phi}_2 &= \Phi_2 - \Phi_1, & \hat{\Phi}'_2(Z') &= \hat{\Phi}_2 - \Lambda - \bar{\Lambda},\end{aligned}\tag{16}$$

where on the r.h.s. of (16) the superconformal transformations of the redefined superfields are reproduced. In terms of  $\hat{\Phi}_\alpha$  Eqs. (13), (14) take the form <sup>3</sup>:

$$\begin{aligned}\bar{D}D\hat{\Phi}_2 &= ie^{\hat{\Phi}_2}(1 + ie^{3\hat{\Phi}_1 - 2\hat{\Phi}_2}\bar{\Psi}^{+b}\Psi^{-b}), \\ \bar{D}D\hat{\Phi}_1 &= \frac{1}{3}e^{3\hat{\Phi}_1 - \hat{\Phi}_2}\bar{\Psi}^{+b}\Psi^{-b}, & \bar{D}\Psi^{-b} &= 0 = D\bar{\Psi}^{+b}. \\ D\Psi^{-b} + 2D\hat{\Phi}_1\Psi^{-b} &= 1, & \bar{D}\bar{\Psi}^{+b} + 2\bar{D}\hat{\Phi}_1\bar{\Psi}^{+b} &= 1.\end{aligned}\tag{17}$$

The constraints (18) are explicitly solvable [10, 8] and result in the following form of  $\Psi^{-b}$ ,  $\bar{\Psi}^{+b}$  and  $\hat{\Phi}_1$ :

$$\begin{aligned}\Psi^{-b} &= c(z) + \theta(1 + ic\partial c), & \bar{\Psi}^{+b} &= \bar{c}(\bar{z}) + \bar{\theta}(1 + i\bar{c}\bar{\partial}\bar{c}), \\ \hat{\Phi}_1 &= \hat{\phi}_1(z, \bar{z}) + \frac{i}{2}\theta e^{-2\hat{\phi}_1}\partial(e^{2\hat{\phi}_1}c) + \frac{i}{2}\bar{\theta}e^{-2\hat{\phi}_1}\bar{\partial}(e^{2\hat{\phi}_1}\bar{c}) + \theta\bar{\theta}c\bar{c}\partial\bar{\partial}\hat{\phi}_1.\end{aligned}\tag{19}$$

Note that the superconformal symmetry is spontaneously broken since under the superconformal transformations (4), (15) (with  $z \rightarrow z - \lambda(z) - i\theta\epsilon$ ,  $\bar{z} \rightarrow \bar{z} - \bar{\lambda}(\bar{z}) - i\bar{\theta}\bar{\epsilon}$ ,  $\theta \rightarrow \theta - \epsilon(z) - \theta\partial\lambda$ ,  $\bar{\theta} \rightarrow \bar{\theta} - \bar{\epsilon}(\bar{z}) - \bar{\theta}\bar{\partial}\bar{\lambda}$  and  $\Lambda(Z) = -\partial\lambda - i\theta\partial\epsilon$ ) the spinor fields  $c(z)$  and  $\bar{c}(\bar{z})$  transform as Goldstone fermions of conformal spin  $-\frac{1}{2}$ :

$$\begin{aligned}\delta_\lambda c &= \lambda\partial c - \frac{1}{2}c\partial\lambda, & \delta_\epsilon c &= c(z) + \epsilon(z) + i\epsilon(z)c\partial c, \\ \delta_{\bar{\lambda}}\bar{c} &= \bar{\lambda}\bar{\partial}\bar{c} - \frac{1}{2}\bar{c}\bar{\partial}\bar{\lambda}, & \delta_{\bar{\epsilon}}\bar{c} &= \bar{c}(\bar{z}) + \bar{\epsilon}(\bar{z}) + i\bar{\epsilon}(\bar{z})\bar{c}\bar{\partial}\bar{c}.\end{aligned}\tag{20}$$

Finally we present the component form of Eqs. (17), where  $\phi_2 = \hat{\Phi}_2|_{\theta, \bar{\theta}=0}$ ,  $\psi = iD\hat{\Phi}_2|_{\theta, \bar{\theta}=0}$ ,  $\bar{\psi} = i\bar{D}\hat{\Phi}_2|_{\theta, \bar{\theta}=0}$  and  $\phi_1 = 3\hat{\phi}_1 - ic(\frac{1}{2}\partial c + \psi) - i\bar{c}(\frac{1}{2}\bar{\partial}\bar{c} + \bar{\psi})$  (this  $\phi_1$  should not be confused with the leading component of  $\Phi_1$  in (13)):

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<sup>3</sup>In what follows we put  $\nu_\alpha = \mu_\alpha = 1$  without loosing generality



$$\begin{aligned}
\bar{\partial}\partial\phi_1 &= -e^{\phi_1-\phi_2} - ie^{\phi_1}\bar{c}c - i\partial(e^{\phi_2}c\bar{\psi}) + i\bar{\partial}(e^{\phi_2}\bar{c}\psi), \\
\bar{\partial}\partial\phi_2 &= -e^{\phi_2}(e^{\phi_2} + i\bar{\psi}\psi) + e^{\phi_1-\phi_2}, \\
\bar{\partial}\psi &= e^{\phi_2}\bar{\psi} - e^{\phi_1-\phi_2}c, \quad \partial\bar{\psi} = -e^{\phi_2}\psi - e^{\phi_1-\phi_2}\bar{c}, \quad \bar{\partial}c = 0, \quad \partial\bar{c} = 0.
\end{aligned} \tag{21}$$

From Eqs. (21) it follows that the free Goldstone fermions  $c(z)$ ,  $\bar{c}(\bar{z})$  contribute to the r.h.s. of equations of motion of  $\phi_1$ ,  $\psi$  and  $\bar{\psi}$ . This is in contrast to the simplest case of the Hamiltonian reduction of this kind applied to the affine superalgebra  $sl(2, \mathbf{R})$ , which leads to the super-Liouville system with completely decoupled bosonic and fermionic sector [8].

## 4 Constraints and symmetries of the model

Let us now turn to the consideration of the Hamiltonian properties of the constraints (11), the resulting number of independent fields of the model and their symmetry properties.

One can directly check that all the constraints (11) commute with each other with respect to the Poisson brackets (6) except for the Grassmann component of  $J^{-b} - \mu_b = 0$  (and  $\bar{J}^{+b} - \nu_b = 0$  in the antiholomorphic sector) whose Poisson bracket with itself produces the  $\delta$ -function on the r.h.s. of (6). Thus all the constraints (11) are of the first class except this fermionic second-class constraint. Analogous situation encountered in the  $Sl(2, \mathbf{R})$  case has been described in detail in [8].

The first-class constraints generate local gauge transformations of the  $OSp(1|4)$  supercurrent components that have not been constrained. This allows one to impose on some of these currents gauge conditions. The number of the gauge conditions is equal to the number of the first-class constraints. This reduces the initial number of the supercurrent components to an independent set of bosonic and fermionic current fields. In the case under consideration we initially have  $14 \times 2 = 28$  bosonic and fermionic fields. The  $5 \times 2 = 10$  first-class constraints ( $\Psi^{-i} = 0, (i = 1, \dots, 4), \Psi^{-f} = \mu_f$ ) together with corresponding gauge conditions eliminate  $20 = 10 \times 2$  degrees of freedom, while the constraint  $J^{-b} = \mu_b$  (which is a mixture of the first and second class constraint) eliminates 2 bosonic and 1 fermionic degree of freedom.

One can gauge fix to zero those  $OSp(1|4)$ -valued supercurrents whose Poisson brackets [2, 3, 4] with one of the first-class constraints contains a  $\delta$ -function term. This indicates that under the corresponding gauge transformation the supercurrent undergoes an arbitrary shift proportional to the parameter of the transformation and hence can be eliminated. This way one can put to zero the following components of (2) and (10) (in the holomorphic sector):

$$\Psi^{0f} = \Psi^{+f} = \Psi^{+b} = J^{0b}|_{\theta=0} = J^{+2} = J^{+3} = 0.$$

Note that despite of the non-manifestly supersymmetric form the constraint  $J^{0b}|_{\theta=0} = 0$  is superconformal invariant (see [8]).

Thus, we remain with  $28 - 23 = 5$  independent currents both in the holomorphic and in the antiholomorphic sector of the constrained model. Among them we have 2 fermionic degrees of freedom corresponding to fields entering the fermionic supercurrents  $\Psi^{+1}, \bar{\Psi}^{-1}$  (associated with the  $OSp(1|4)$  non-simple roots  $E_{\pm 1}$ ). These two chiral and antichiral fermionic fields (which we call  $b(z)$  and  $\bar{b}(\bar{z})$ ) have conformal spin  $\frac{3}{2}$ . By an appropriate change of variables they can be made canonical conjugate to the spin  $(-\frac{1}{2})$  fields  $c(z), \bar{c}(\bar{z})$  with respect to Dirac brackets which one can construct by use of the constraints and the gauge conditions (see, for instance [8]).

Since  $\Psi^{+1}$  and  $\bar{\Psi}^{-1}$  do not enter the super-Toda equations (13, 14), the super-Toda system obtained in the previous Section does not contain all fields of the initial WZNW model which survive the Hamiltonian reduction. It is a closed superconformal subsystem of interacting fields which is part of a larger system which also includes a completely decoupled pair of free chiral fermions of conformal spin  $\frac{3}{2}$ . This is a consequence of the presence in (11) of the second-class constraints, and is in contrast to the standard Hamiltonian reduction of bosonic Lie algebras and superalgebras with entirely fermionic simple root structure where the Cartan subalgebra fields of corresponding (super)-Toda model exhaust the number of remaining physical degrees of freedom [2, 3, 4].

In the holomorphic (as well as in the antiholomorphic) sector the above system of constraints and gauge fixing conditions is explicitly solved in terms of five unconstrained fields having conformal spins

$$\left(-\frac{1}{2}, \frac{3}{2}, 2, \frac{5}{2}, 4\right).$$

Let us call them

$$(c, G, T, W_{\frac{5}{2}}, W_4). \quad (22)$$

On the one hand these fields are a combination of  $OSp(1|4)$  supercurrent components and on the other hand they are expressed in terms of  $\hat{\phi}_1, \phi_2, \psi, b, c$  and their derivatives.

The conformal spin 2 current  $T(z)$  and the conformal spin  $\frac{3}{2}$  current  $G(Z)$  are given by:

$$\begin{aligned} T &= T_1 + T_2 - \frac{3i}{2}b\partial c - \frac{i}{2}\partial bc, \\ T_1 &= k \left( (\partial\hat{\phi}_1)^2 - \partial^2\hat{\phi}_1 \right), \quad T_2 = k \left( (\partial\phi_2)^2 - \partial^2\phi_2 + \frac{i}{2}\psi\partial\psi \right); \\ G &= G_1 + G_2; \end{aligned} \quad (23)$$

$$G_1 = ib + icT_1 - bc\partial c - \frac{k}{4}c\partial c\partial^2 c - ik\partial^2 c, \quad G_2 = ik(\psi\partial\phi_2 - i\partial\psi).$$

They generate on the Dirac brackets the classical  $n=1$  super-Virasoro algebra with central charge  $c_{tot} = 12k$ , where  $k$  is a level [2, 3, 4] of affine  $OSp(1|4)$  and  $c_{tot}$  is the sum of the two central charges  $c_1, c_2$  related to the two independent super-Virasoro realizations  $(T_1, G_1)$  and  $(T_2, G_2)$ .

The realization (23) of the superconformal algebra involves the  $(b - c)$  system which plays essential role in various physical applications of superconformal theory (see, for example, [5, 12, 13, 8]).

The fields  $W_{\frac{5}{2}}$  and  $W_4$  in (22) are primary fields with conformal spin  $\frac{5}{2}$  and 4, respectively. The derivation of their form in terms of the super-Toda fields and  $b(z)$  is pretty cumbersome and we have not carried out it explicitly. Thus we have not yet checked that the full set of the fields (22) generate on the Dirac brackets a closed classical super- $\mathcal{W}$  algebra, though the consistency of the Hamiltonian reduction procedure performed and the relation of our model with that based on  $OSp(1|2)$ ,  $Sp(4)$  and  $SL(2, \mathbf{R})$  group (as discussed in section 3) suggests possible existence of such an algebra. If this is indeed the case, such a new super- $\mathcal{W}$  algebra should be a supersymmetric extension of the so-called  $WB_2$  algebra [17] generated by a stress energy tensor and two primary fields (analogous to  $W_{\frac{5}{2}}$  and  $W_4$ ) of conformal spin  $\frac{5}{2}$  and 4. Indeed, in [16] the  $WB_2$  algebra was obtained by imposing constraints on affine  $OSp(1|4)$  currents which were taken to be ordinary two-dimensional fields and not superfields as in our case. For that reason the Hamiltonian reduction of affine  $OSp(1|4)$  carried out in [16] resulted in a non-supersymmetric  $\mathcal{W}$ -structure, and its supersymmetric extension might be generated by (22).

In conclusion we have obtained the supersymmetric Toda model with peculiar symmetry properties by applying the generalized Hamiltonian reduction procedure to the affine  $OSp(1|4)$  superalgebra. From the physical point of view it seems of interest to study, within the frame of a geometrical approach [18, 10, 8, 19], whether the model considered herein describes the dynamics of a superstring propagating in  $D = 4$  anti-De-Sitter superspace whose symmetry group is  $OSp(1|4)$ .

**Acknowledgements.** The authors would like to thank E. Ivanov, S. Krivonos, Z. Popovicz, A. Sorin and M. Vasiliev for interest to this work and useful discussion. Work of D.S. was partially support by the INTAS grant N 93-493.

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